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Why, how, and when, MHD turbulence becomes two-dimensional

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A description of MHD turbulence at low magnetic Reynolds number and large interaction parameter is proposed, in which attention is focussed on the role of insulating walls perpendicular to a uniform applied magnetic field. The flow is divided in two regions: the thin Hartmann layers near the walls, and the bulk of the flow. In the latter region, a kind of electromagnetic diffusion along the magnetic field lines (a degenerate form of Alfvén waves) is displayed, which elongates the turbulent eddies in the field direction, but is not sufficient to generate a two-dimensional dynamics. However the normal derivative of velocity must be zero (to leading order) at the boundaries of the bulk region (as at a free surface), so that when the length scale $l_\perp$ perpendicular to the magnetic field is large enough, the corresponding eddies are necessarily two-dimensional. Furthermore, if $l_\perp$ is not larger than a second limit, the Hartmann braking effect is negligible and the dynamics of these eddies is described by the ordinary Navier–Stokes equations without electromagnetic forces. MHD then appears to offer a means of achieving experiments on two-dimensional turbulence, and of deducing velocity and vorticity from measurements of electric field.

1. Introduction

There is increasing interest in the study of two-dimensional turbulence. Indeed, it is thought to provide a good schema for large scale motions of geophysical fluids, and perhaps for coherent structures recently observed in mixing layers. Contradictory predictions have been made about the rather singular properties of two-dimensional homogeneous and isotropic turbulence, and direct numerical simulations are not sufficiently reliable to decide between them. It is therefore clear that laboratory observations are of crucial interest. But any attempt is confronted with the necessity of inhibiting the natural instability of two-dimensional structures which quickly spread out into three dimensions.

Body forces such as Coriolis forces in rotating fluids and electromagnetic forces in liquid metals moving under the influence of a uniform magnetic field have been recognized as good candidates for many years (Lehnert 1955). Indeed, both are strictly balanced by pressure gradients in any two dimensional flow (in a plane perpendicular to the magnetic field or the rotation vector) and have some stabilizing effect on three-dimensional instabilities. However, the Coriolis forces do not change the total energy but transfer it by inertial waves while the electromagnetic forces are purely dissipative in laboratory conditions where the magnetic Reynolds number is very small. So the effect of electromagnetic forces on the three-dimensional perturbations is easier to deal with. While two-dimensional turbulence is effectively observed in rotating tanks of small depth (Colin de Verdiere 1980) important three-dimensional perturbations arise in deeper tanks (Hopfinger & Browand 1982). A similar behaviour occurs for turbulent flows in a strong uniform magnetic field, as becomes apparent from examining the two kinds of experiments which have been performed during the last ten years:

(*a*) In rectangular insulating duct flows placed in a strong enough transverse magnetic field, numerous characteristics of two-dimensional turbulence have been observed. The source of turbulence does not seem to have a preponderant influence; whether this is the M-profile instability, or the wake instability behind a grid or behind an array of cylinders parallel to the magnetic field, results such as summarized by Branover (1978) or by Lielausis (1975), seem to be quite general and show that

(i) Velocity correlations along the magnetic field direction are very good over the whole channel width (Votsish & Kolesnikov 1976);

(ii) The velocity component parallel to the magnetic field is much smaller than the other ones, as indirectly shown by turbulent diffusion of a contaminant (Kolesnikov & Tsinober 1974, Sommeria 1980);

(iii) The energy decay is very slow while the integral scale of the turbulence grows rather fast (for example Votsish & Kolesnikov 1976).

(*b*) On the other hand, in experiments reported by Alemany *et al.* (1979) where turbulence is studied behind a grid moving in an axial magnetic field (produced in a solenoid) the dynamics was shown to be quite different:

(i) The integral length scales in the field direction increases rather slowly when the field is increased; the increase of the perpendicular length scales seems limited by the square root of the interaction parameter;

(ii) The three velocity components are of the same order of magnitude;

(iii) Most noticeably, the energy decay is very fast, much faster than without magnetic field in the same facility ($t^{-1\cdot 7}$ instead of $t^{-1\cdot 2}$). This is a consequence of the Joule effect and reveals important departures from two-dimensionality.

The apparent discrepancy between these two kinds of results have led to controversial ideas about laboratory MHD turbulence and one of the purposes of this paper is to reconcile the two. Theories of homogeneous MHD turbulence in a uniform magnetic field, using dimensional analysis and two point closures (Alemany *et al.* 1979) predict a three-dimensional dynamics and agree with the latter kind of experimental results. We intend here to explain the two-dimensional dynamics observed in duct flows in terms of the effect of insulating walls. It is clear that the walls should have an important direct effect on turbulent structures, the length scale of which is equal to or larger than the channel width. The point is that turbulent structures are markedly elongated in the magnetic field direction, when the field is strong, so that the direct effect of the walls which are perpendicular to the field becomes important, even when the grid mesh is much smaller than the channel width. This direct effect is dramatic and rather trivial when the walls are highly conducting: in this case turbulence has been observed to be very quickly damped in a strong magnetic field (Platnieks & Freibergs 1972). So we are interested here in the case of insulating walls when turbulence persists over a large distance and exhibits some two-dimensionality. Note that the effect of walls *parallel* to the magnetic field, remains restricted to side layers which are passive in character and do not react back upon the bulk flow, we shall not consider the effect of such walls.

We intend then in this paper to analyse the direct effect of the walls which are perpendicular to the field on a turbulent structure as a function of its length and velocity scales. The analysis is limited to the asymptotic case of strong magnetic fields (large interaction parameter and Hartmann number) in which evidence of two-dimensional turbulence has been experimentally obtained. The final result is presented in § 4, where the appropriate conditions for getting a wide range of two-dimensional turbulent structures are discussed. For this investigation, the flow is divided into two very different parts: the Hartmann boundary layers and the bulk of the flow. In § 3, concerned with the bulk of the flow, we point out a diffusion mechanism along magnetic field lines, through which motions in planes perpendicular to the field are correlated. These considerations lend some physical insight to the turbulence dynamics and lead to conclusions closely related to those of Alemany *et al.* (1979). A study of the Hartmann layers is carried out in § 3, resulting in boundary conditions for the bulk of the turbulent flow, from which are deduced conditions for the existence of two-dimensional turbulence. The complete flow is in fact always three-dimensional because of the no slip condition at the walls; as a consequence, there is a Hartmann braking effect for the two dimensional bulk of the flow, which is also calculated in § 3.

2. Some properties of MHD turbulent flows at large interaction parameter

An incompressible and electrically conducting fluid (e.g. a liquid metal) is assumed to be in turbulent motion in a uniform magnetic field $\mathbf{B}$. Let us make $l_\perp$ and U the typical length-scale and corresponding velocity for a turbulent eddy in the directions perpendicular to the field (the *transverse directions* for conciseness). The following conditions are assumed to be valid over a wide range of length scales including energy containing ones.

$$\begin{cases} Re = Ul_\perp/\nu \gg 1 \\ Rm = \mu\sigma l_\perp U \ll 1 \end{cases} \quad \begin{cases} N = \sigma B^2 l_\perp/\rho U \gg 1 \\ M = (N\,Re)^{\frac{1}{2}} \gg 1 \end{cases} \tag{1}$$

where μ, ρ, σ, ν stand for magnetic permeability, density, electrical conductivity and kinematic viscosity respectively. Quantities of interest such as velocity $\mathbf{v}$, current density $\mathbf{j}$, pressure $\tilde{p}$, and electric potential ϕ, then satisfy the following equations (where $d/dt = \partial/\partial t + \mathbf{v}.\nabla$):

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla\tilde{p} + \nu\nabla^2\mathbf{v} + \frac{1}{\rho}\mathbf{j}\times\mathbf{B}, \tag{2}$$

$$\nabla.\mathbf{v} = \nabla.\mathbf{j} = 0, \tag{3}$$

$$\mathbf{j} = \sigma(\mathbf{v}\times\mathbf{B} - \nabla\phi). \tag{4}$$

Notice that if the magnetic Reynolds number Rm is very small, a condition satisfied in most laboratory experiments, the fluctuating magnetic field created by eddy current density is negligible in comparison with $\mathbf{B}$. By eliminating current density, and including the irrotational part of the Lorentz force in the pressure p, the equation of motion becomes (Roberts 1967, p. 136).

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} - \frac{\sigma B^2}{\rho}\Delta^{-1}\frac{\partial^2\mathbf{v}}{dz^2}, \tag{5}$$

where Δ^{-1} is an inverse of the Laplacian operator, and where the z-co-ordinate is taken along the direction of the magnetic field (*the parallel direction*).

Under the conditions (1), the energy-containing eddies are rapidly lengthened in the **B** direction even if the turbulence was initially isotropic (Alemany *et al.* 1979) and the following approximations are valid.

$$\frac{\partial}{\partial z} \ll \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \quad v_z \lesssim v_x, v_y.$$

Using gradient $\nabla_\perp$ and Laplacian $\Delta_\perp$ operators in a transverse plane and substantive derivative associated with the motion in that plane ($D/Dt = \partial/\partial t + \mathbf{v}_\perp . \nabla_\perp$), the equations of motion then become:

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho}\nabla_\perp p + \nu\Delta_\perp v - \frac{\sigma B^2}{\rho}\Delta_\perp^{-1}\frac{\partial^2 \mathbf{v}}{\partial z^2}, \tag{6}$$

$$\nabla_\perp . \mathbf{v}_\perp = 0. \tag{7}$$

The parallel component of vorticity ω_z (but not any other component) satisfies:

$$\frac{D\omega_z}{Dt} = \nu\Delta_\perp \omega_z - \frac{\sigma B^2}{\rho}\Delta_\perp^{-1}\frac{\partial^2 \omega_z}{\partial z^2}. \tag{8}$$

Notice that in this kinematically quasi-two-dimensional situation, motions in different transverse planes interact only through electromagnetic forces and viscosity, but the latter is much weaker when the Reynolds number is large. If we refer to a given eddy for which the application of the $\Delta_\perp^{-1}$ operator reduces to multiplication by $(-l_\perp^2)$, the electromagnetic force looks like a unidirectional diffusion term. This diffusion in the field direction may be seen as a relic of Alfvén wave propagation when $Rm \ll 1$ (Roberts 1967, p. 137). It has the unusual property of being characterized by a diffusivity $\alpha = \sigma B^2 l_\perp^2/\rho$ depending on the length scale $l_\perp$ of the eddy considered. As a consequence, electromagnetic forces tend to suppress velocity differences between transverse planes and, if d denotes their spacing, the duration of this phenomenon is

$$t_d \simeq (\rho/\sigma B^2)\, d^2/l^2. \tag{9}$$

If the turbulence is homogeneous, the net effect of this mechanism is Joule dissipation of energy, as analysed by Alemany *et al.* (1979) in Fourier space. But it is remarkable that however globally dissipating, this diffusion mechanism may be *locally* a source of momentum, vorticity and kinetic energy, and this could explain how two-dimensional turbulent eddies build up from initially three-dimensional ones.

Figure 1 shows a plausible picture of the deformation of a turbulent structure of initial typical size l during its turnover time t_{tu}. Nonlinear transfer would by itself rotate the eddy and appreciably promote its disruption into smaller eddies (size $\simeq \frac{1}{2}l$), but electromagnetic diffusion propagates this motion all along a cylinder of length $l_\parallel \sim l_\perp N^{\frac{1}{2}}$ during time t_{tu}. The main consequences of this are:

(i) inhibition of energy transfer towards small scales and some slight increase of the eddy transverse size which is quite different from the inverse cascade of energy in two-dimensional turbulence.

(ii) the attainment after some transitory phase of an anisotropic state in which, for each turbulent structure,

$$l_\parallel/l_\perp \sim N^{\frac{1}{2}} \tag{10}$$

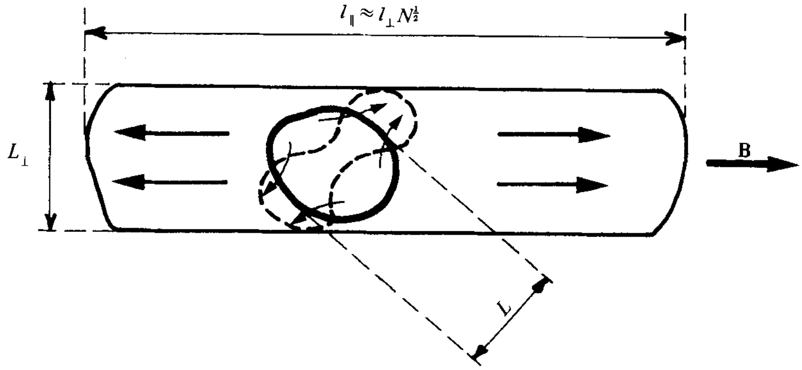

FIGURE 1. Plausible evolution of an eddy of initial size L in all directions during time t_{tu} for an observer moving with the core of the eddy (—— initial eddy; - - - - - final eddy when $\mathbf{B} = 0$; —— final eddy when $N \gg 1$).

(remember that N is the interaction parameter for a given eddy). Of course this last conclusion applies only to structures for which $l_{\perp} N^{\frac{1}{2}}$ is smaller than the spacing of the walls perpendicular to the magnetic field.

3. The influence of transverse insulating walls

The key point which elucidates the influence of the transverse walls on the turbulent bulk of the flow, when the interaction parameter N and the Hartmann number M are both very large, is the fact that inertia is small, compared to electromagnetic and viscous effects in the Hartmann layer. This essentially follows from the extremely small thickness of this layer

$$\delta_H = (\rho\nu/\sigma B^2)^{\frac{1}{2}} = l_{\perp}/M,$$

which depends only on the fluid properties and the strength of the magnetic field ($\delta_H = 30\,\mu$m in mercury for $B = 1$ T). As a consequence the equations of motion in the Hartmann layer are linear to a good approximation, and the flow in this layer depends at a given time only on the instantaneous outer velocity (e.g. secondary flows and inertia effects are negligible in the layer). In other words, the Hartmann layer behaves as if the outer flow were locally steady and uniform in transverse directions.

To be more accurate, the small effects of unsteadiness and inhomogeneity of the outer flow can be taken into account by means of a second-order Hartmann theory, based on developments in powers of the two small parameters N^{-1} and M^{-2}. This straightforward but heavy calculation, which is summarized in the appendix, precisely justifies the linear approximation when N and M are both much larger than unity.

Numerous experimental results also support this linear approximation and confirm the stability of this thin Hartmann layer. For example the experimental skin friction in ducts is given by laminar laws, despite the presence of turbulence. More direct confirmation is achieved in the simple educational experiment of Heiser & Shercliff (1965) or in the annular flow of Gel'fgat *et al.* (1971).

The Hartmann layer along an insulating wall may be seen as a current sheet, the current content $\mathbf{J}$ per unit length of which is given (Heiser & Shercliff 1965) by

$$\mathbf{J} = (\sigma\rho\nu)^{\frac{1}{2}} \boldsymbol{\beta} \times \mathbf{v}(x, y, 0, t), \tag{11}$$

where $\mathbf{v}(x, y, 0, t)$ is the velocity near the wall (but outside the Hartmann layer). The origin of the z axis is taken at the wall and its positive unit vector $\boldsymbol{\beta}$ is directed toward the fluid. The following important properties follow directly from the relation (11).

(a) *Orthogonality of the quasi two-dimensional eddies with the walls*

Because of current conservation, a current density j_z necessarily exists outside the Hartmann layer, such that (since the wall is insulating).

$$j_z(x, y, 0, t) = -\nabla_\perp . \mathbf{J} = (\sigma\rho\nu)^{\frac{1}{2}}\,\omega_z(x, y, 0, t) \tag{12}$$

where ω_z is the component of vorticity. Alternatively, taking the curl of Ohm's law twice yields

$$\Delta\mathbf{j} = \sigma B\,\partial\boldsymbol{\omega}/\partial z. \tag{13}$$

To be consistent with previous approximations leading to equations (6) and (7), we have to replace Δ by $\Delta_\perp$. Now eliminating j_z between (12) and the z component of (13), the electrical boundary conditions at an insulating wall can be transformed into a single condition on transverse vorticity,

$$\frac{\partial\omega_z}{\partial z}(x, y, 0, t) = -\frac{l_\perp}{M}\Delta_\perp\,\omega_z(x, y, 0, t). \tag{14}$$

Since this relation is valid at any position (x, y) along the wall, it necessarily requires that

$$\frac{\partial v_\perp}{\partial z}(x, y, 0, t) = O\left(\frac{U}{Ml_\perp}\right) = O\left(\frac{U}{\sqrt{Re}\,l_\parallel}\right), \tag{15}$$

where $l_\parallel$ is the natural parallel length scale introduced by relation (10). Since $U/l_\parallel$ would be a typical value of $\partial v_\perp/\partial z$ in the absence of wall effects, the order of magnitude (15) means that axes of quasi-two-dimensional eddies have their ends perpendicular to the walls to a good approximation, although they could bend in the bulk of the flow. This condition can be physically interpreted using the notion of electromagnetic diffusion introduced in § 2: there is a negligible flux of momentum through the insulating walls.

(b) *Hartmann braking of the outer turbulence*

The current sheet in the Hartmann layer, which gives rise to the current density $j_z(x, y, 0, t)$ (12), must necessarily be closed in the bulk of the flow, generating there some braking and thus transferring the influence of viscosity. More precisely, when the walls are everywhere insulating, current conservation requires that:

$$\nabla_\perp \cdot \int_0^a \mathbf{j}_\perp\,dz = 0, \tag{16}$$

where a is the spacing between the walls; this means that no external source of current exists, and reduces to the usual condition

$$\int_0^a \mathbf{j}_\perp\,dz = 0$$

if the outer flow is uniform. The integral in (16) is composed of two parts, the current sheets $\mathbf{J}(0)$ and $\mathbf{J}(a)$ in the Hartmann layers, and the current integrated in the bulk of the flow. These are both related to velocity, the former by (11), and the latter by the equation of motion (2). After these substitutions, the relation (16) takes the form:

$$\frac{\rho a}{B}\left\langle\frac{d\omega_z}{dt}-\nu\nabla^2\omega_z\right\rangle+(\sigma\rho\nu)^{\frac{1}{2}}\,(\omega_z(a)+\omega_z(0)) = 0, \tag{17}$$

where the brackets $\langle\ \rangle$ indicate the average between $z = 0$ and $z = a$.

Equation (17) clearly demonstrates the braking of the outer flow by the Hartmann effect: the last term on the left-hand-side is exactly the loss of vorticity due to the viscous stresses in the Hartmann layer, which is globally transferred into the outer turbulence by the return of electric currents. If two-dimensionality is well achieved the brackets in (17) can be omitted (§ 4 shows how and when this is realized) and the equation of motion for the two-dimensional velocity field $\mathbf{V}\ (x, y, t)$ can be derived:

$$\frac{d\mathbf{V}}{dt}=\frac{1}{\rho}\nabla P+\nu\nabla^2\mathbf{V}-\frac{2B}{a}\left(\frac{\sigma\nu}{\rho}\right)^{\frac{1}{2}}\mathbf{V}. \tag{18}$$

It is finally of interest to note that this braking does not depend on the turbulent scales, provided that M (constructed from these scales) is very large, its typical time-scale being

$$t_H=\frac{a}{2B}\left(\frac{\rho}{\sigma\nu}\right)^{\frac{1}{2}}. \tag{19}$$

(c) *Interpretation of electric field measurements*

Measurements of electric field, including the fluctuating part, have often been used to get information on turbulent flows (Lielausis 1975, Rosant 1976, Moreau 1978). The main idea is that in the presence of a magnetic field B the turbulent structures of typical scales U and $l_\perp$ are submitted to potential differences of which a representative value is $\phi \sim BUl_\perp$ and which generate parallel electric currents (essentially related to the defect of two-dimensionality):

$$j_z \simeq \sigma\phi/l_\parallel \simeq \sigma BU/N^{\frac{1}{2}}, \tag{20}$$

and transverse electric currents, which follow from current conservation

$$j \simeq \sigma BU(l_\perp/l_\parallel) \simeq \sigma BU/N. \tag{21}$$

These orders of magnitude mean that the relation $\mathbf{E}_\perp = -\mathbf{v}\times\mathbf{B}$ is satisfied with an error of the order of N^{-1}, and then justify deduction of the turbulent velocity from the fluctuations of the electric field when the magnetic field is strong enough.

Now, in the turbulent core between two insulating transverse walls, the electric current can be calculated precisely by condition (16) together with relation (11). When two-dimensionality is well achieved, using Ohm's law yields the expression for the transverse electric field:

$$\mathbf{E}_\perp = -(1-2l_\perp/aM)\,(\mathbf{V}\times\mathbf{B}). \tag{22}$$

Since $l_\perp/M$ does not depend on $l_\perp$, this expression establishes the required connection between $\mathbf{E}_\perp$ and $\mathbf{V}$.

From relation (22) and the equation of continuity of electric current with the boundary conditions (12), an expression for E_z in terms of the vorticity Ω can be deduced, viz.

$$E_z=\left(\frac{\rho\nu}{\sigma}\right)^{\frac{1}{2}}\Omega\left(1-\frac{2z}{a}\right). \tag{23}$$

An original method of vorticity measurement for the two-dimensional flow is thus provided by this result. Nevertheless, the order of magnitude of the corresponding potential differences is generally very small. For example in mercury, if the two electrodes measuring the electric field are 1 cm distant, and the turbulent scales $U = 1$ cm/s and $l_\perp = 1$ cm, the potential difference is of order of $0\cdot3$ μV, thus at the very limit of experimental detectability. But some expedients should be used to increase this value. One could artificially increase the Hartmann layer thickness by putting some controlled rugosity at the wall near the probe. One could also fix a conducting sheet on the wall to increase the current content. If σ' is the conductivity and d the thickness of the sheet, the electric field would then become

$$E_z = \left[\left(\frac{\rho\nu}{\sigma}\right)^{\frac{1}{2}} + \frac{\sigma'}{\sigma}d\right]\Omega_z\left(1-\frac{2z}{a}\right).$$

4. The behaviour of MHD turbulence between insulating walls

The purpose of this section is to deduce from the relations (15) and (18) the main rules governing the dynamics of a turbulent velocity field when the interaction parameter is large. Though the analysis is developed in the particular case of turbulence which is assumed to be homogeneous in a plane perpendicular to **B**, it has nevertheless some relevance for other kinds of turbulent flows, such as duct flows with or without M-shaped mean velocity profiles.

It follows from Alemany *et al.* (1979) that the dynamics of MHD homogeneous turbulence is essentially governed by an equilibrium between two competing mechanisms: Joule effect which tends to dissipate energy of wave vectors not perpendicular to the applied magnetic field, and inertial transfers which tend to restore isotropy. This description is quite in agreement with precise calculations using a two-point closure technique, which show that angular energy transfer (from the energy-containing zone to the cones where Joule dissipation is predominant) is much larger than radial energy transfers, and is localized in a narrow fringe at the border of the Joule cones. Figure 2 (left-hand side) illustrates this description. The energy-containing zone is characterized by an angle ϕ such that the Joule time t_j and the eddy turnover time t_{tu} are equal ($\phi \approx N^{-\frac{1}{2}}$, if $N \gg 1$). Alemany *et al.* deduced from this quasi-steady equilibrium that, since the local transfer time $[k^3E(k,t)]^{-\frac{1}{2}}$ must be independent of k, just as the Joule time is, the energy spectrum $E(k,t)$ must vary like k^{-3} in a self-similar range. Furthermore, the energy equation ($\partial E/\partial t \simeq -E(k^3E)^{\frac{1}{2}}$ demands that the energy spectrum decays as t^{-2}. The properties of this self-similar range with $E(k,t) \simeq t^{-2}k^{-3}$ are fairly well confirmed by experiments and two-point closure calculations.

Now the influence of the insulating walls perpendicular to the magnetic field has to be introduced in this context. It is first expressed by the constraint (15) which means that the parallel half wavelength of the energy-containing eddies cannot vary continuously, but must be either infinite (for exactly two-dimensional eddies) or an integer fraction a/n of the duct width (n is an integer). In other words energy must be concentrated in wave vectors with their extremities on a set of planes such that $k_z = n\pi/a$ (figure 2). Otherwise, except possibly during an initial phase, energy-containing vectors can only lie on the part of this set of planes which is outside the Joule cone: $k_z < k_\perp N^{-\frac{1}{2}}$. Thus the first influence of the insulating walls consists in

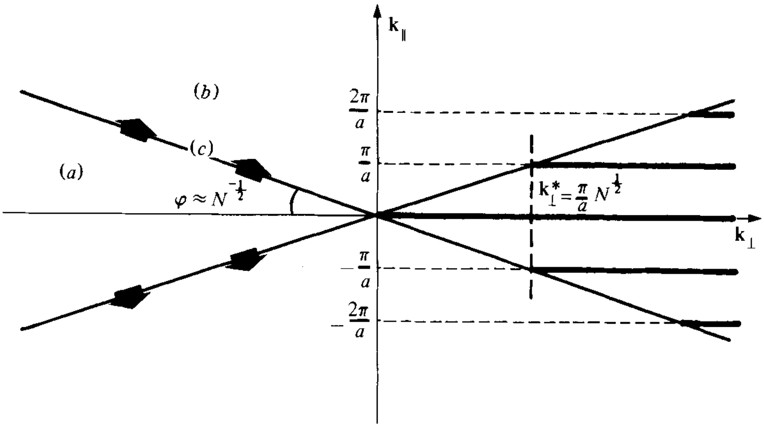

FIGURE 2. Left-hand side: Localization of (*a*) energy-containing, (*b*) dissipating, and (*c*) transferring zones in homogeneous turbulence after Alemany *et al.* (1979). Right-hand side: Quantization of the energy-containing zone (——) when perpendicular walls are insulating.

quantizing the Fourier space, quite an unusual situation in turbulence, which however does not much affect the small scales.

The condition for restricting energy-containing eddies to exactly two-dimensional ones appears now very clearly, namely $k_\perp < (\pi/a) N^{-\frac{1}{2}}$, i.e. since the interaction parameter depends on $k_\perp$,

$$k_\perp < \pi/a(\sigma B^2 a/\rho U)^{\frac{1}{3}}. \tag{24}$$

But, so far, it can only be concluded that these turbulent structures are *kinematically* two-dimensional. First of all their behaviour should differ from that of exactly two-dimensional turbulence in that they support the angular transfer to the dissipative levels $k_z = n\pi/a$. But one is easily convinced that this argument fails. Since inertial transfers require triad interactions, if no energy is present in these dissipative levels, energy transfer to them is also zero. One can also invoke locality of angular energy transfer displayed by Alemany *et al.* (1979) to conclude that this transfer, if any, would only reach the vicinity of the plane $k_z = 0$ (but it is inhibited by the quantization) and would not be able to reach even the first dissipative level $k_z = \pi/a$.

Secondly, the dynamics of these two-dimensional structures is not governed by the ordinary two-dimensional Navier–Stokes equation, but by equation (18) in which Hartmann braking effect is present. Unlike the preceding argument against pure two-dimensionality, this one cannot exactly fail. It can however become a negligible restriction if the Hartmann decay time t_H (19) is much larger than the local turnover time. This condition provides a second limit

$$k_\perp > (B/Ua)(\sigma\nu/\rho)^{\frac{1}{2}}, \tag{25}$$

and one can finally conjecture that a range of wavenumber $k_\perp$ exists which are not only *kinematically*, but also *dynamically* two-dimensional. Since these limits depends on the velocity scale ($U \simeq (k_\perp E(k_\perp))^{\frac{1}{2}}$), it is useful to plot them in a $(k_\perp, E)$ diagram (figure 3). Two other limits ($N = 10$ and $Re = 10$) have also been plotted to indicate the domain of validity of the preceding results.

It is interesting to note that two-dimensional ordinary turbulence also obeys a spectral decay law $E(k,t) \propto t^{-2}k^{-3}$, corresponding to an enstrophy cascade (see

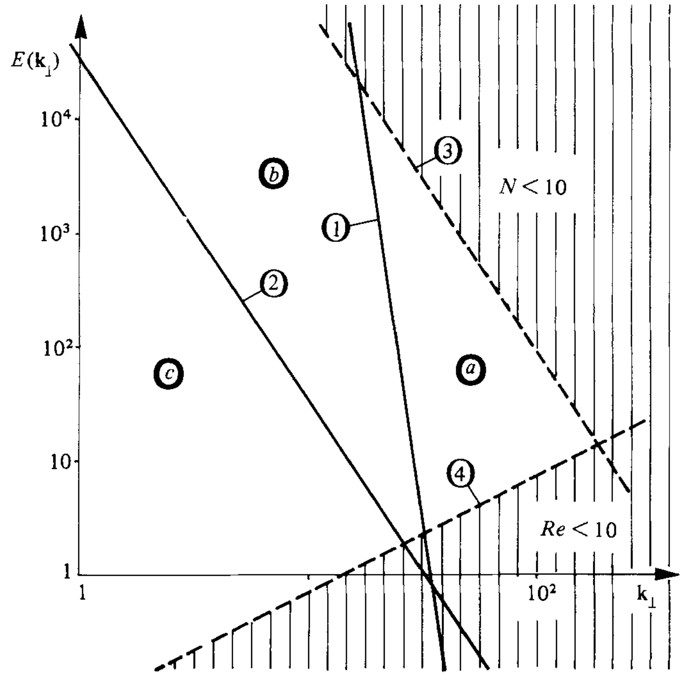

FIGURE 3. The three typical kinds of MHD turbulence between insulating walls perpendicular to a strong uniform magnetic field. ①: Condition (37) for kinematically two-dimensional turbulence; ②: condition (38) for Hartmann braking effect negligible; ③: $N = 10$; ④: $Re = 10$. (a): Equilibrium between angular transfer and Joule effect; (b): Dynamically two-dimensional eddies; (c): two-dimensional eddies with Hartmann effect predominant.

Batchelor 1969, Tatsumi & Yanase 1977). In most of the experiments mentioned in the Introduction, conditions were such that for the largest scales two-dimensionality without the Hartmann braking effect was fairly well achieved (suggesting the relevance of an enstrophy cascade) whereas for smaller scales angular transfer and Joule dissipation give the only reasonable explanation of the observed $t^{-2}k^{-3}$ law. It is noteworthy that no discontinuity between these two laws has been observed. This suggests that the coefficient of $t^{-2}k^{-3}$ could be unique and universal, like the Kolmogoroff and Batchelor constants.

5. Conclusions

MHD turbulence between insulating walls perpendicular to a strong magnetic field ($N \gg 1$) appears to be governed by essentially two mechanisms. One is the *electromagnetic diffusion* into which Alfvén waves degenerate at the scale of the laboratory. The other is the well known *Hartmann effect*, slightly modified by the presence of inertia and vorticity in the external velocity field. Neither of these is new, and their combined influence has been studied in textbooks (e.g. Shercliff 1965, pp. 160–6), but their relevance and importance in turbulence studies has not been recognized previously.

One of the most striking consequences of the relations (14) and (15) is the necessary orthogonality of energy-containing eddies and insulating walls, entailing some quantization of the Fourier space and hence an explanation of the previously observed two-dimensionality. Several conclusions are also quantitative, e.g. equation (18) for the external velocity field, and relations (22) and (23) between electric field and velocity or vorticity.

In the context of the new possibilities offered by MHD for performing experiments on ordinary two-dimensional turbulence, our two main results are (*a*) the formulation of conditions (24) and (25) for the existence of this kind of disordered motion, and (*b*) the recognition that MHD offers interesting new diagnostic techniques. Indeed all the components of the electric field are now precisely related to the velocity field. Measuring the parallel component in the vicinity of the wall would directly give the parallel vorticity, a quantity of prime interest in the context of intermittency studies. And relation (22) shows that the perpendicular components of the electric field are proportional to the velocity.

The authors have benefitted from useful discussions with A. Alemany and M. Lesieur.

Appendix. Second-order calculation of the Hartmann layer

Let us introduce the following dimensionless variables:

$$\left.\begin{array}{lll} x = l_\perp \chi, & v_x = Uu, & \tilde{p} = \rho U^2 P, \\ y = l_\perp \eta, & v_y = Uv, & \phi = BU l_\perp \phi, \\ z = \dfrac{l_\perp}{M}\zeta, & v_z = \dfrac{U}{MN} w, & t = \dfrac{l_\perp}{U}\tau. \end{array}\right\} \tag{26}$$

The length scales are suggested by the discussion of the beginning of § 3 and the order of magnitude of v_z is obtained by considering it as resulting from secondary flows in the Hartmann layer. The basic equations (2)–(4) may now be written:

$$\left.\begin{array}{l} \nabla_\perp . \mathbf{v}_\perp = -\dfrac{1}{N}\dfrac{\partial w}{\partial \zeta}, \\ \dfrac{\partial^2 u}{\partial \zeta^2} - u - \dfrac{\partial \phi}{\partial \eta} = \dfrac{1}{N}\left(\dfrac{du}{d\tau} + \dfrac{\partial P}{\partial \chi}\right) - \dfrac{1}{M^2}\Delta_\perp u, \\ \dfrac{\partial^2 v}{\partial \zeta^2} - v + \dfrac{\partial \phi}{\partial \chi} = \dfrac{1}{N}\left(\dfrac{dv}{d\tau} + \dfrac{\partial P}{\partial \eta}\right) - \dfrac{1}{M^2}\Delta_\perp v, \\ \dfrac{\partial P}{\partial \zeta} = \dfrac{1}{M^2}\dfrac{\partial^2 w}{\partial \zeta^2} - \dfrac{1}{NM^2}\dfrac{dw}{d\tau} + \dfrac{1}{M^4}\Delta_\perp w, \\ \dfrac{\partial^2 \phi}{\partial \zeta^2} = \dfrac{1}{M^2}(\omega - \Delta_\perp \phi), \end{array}\right\} \tag{27}$$

where, for the sake of conciseness, we denote:

$$\begin{array}{ll} d/d\tau = \partial/\partial\tau + u\,\partial/\partial\chi + v\,\partial/\partial\eta + w\,\partial/\partial\zeta & \\ \nabla_\perp = (\partial/\partial\chi, \partial/\partial\eta, 0) & \Delta_\perp = \partial^2/\partial\chi^2 + \partial^2/\partial\eta^2 \\ u_\perp = (u, v, 0) & \omega = \partial v/\partial\chi - \partial u/\partial\eta. \end{array}$$

Limiting attention to the case of insulating walls, the boundary conditions are

$$\begin{cases} u(\chi,\eta,0,\tau) = v(\chi,\eta,0,\tau) = w(\chi,\eta,0,\tau) = 0, \\ \dfrac{\partial\phi}{\partial\zeta}(\chi,\eta,0,\tau) = 0. \end{cases}$$

In the range of strong magnetic fields in which we are interested, the smallness of the two independent parameters $1/N$ and $1/M^2$ permits a double perturbation technique to solve these equations. We therefore write each non-dimensional function q as

$$q = q^{(0)}(\chi,\eta,\zeta,\tau)+\frac{1}{N}q^{(1)}(\chi,\eta,\zeta,\tau)+\frac{1}{M^2}q^{(2)}(\chi,\eta,\zeta,\tau)+\dots .$$

To first order the leading terms, all placed on the left-hand side of equations (27), give the classical Hartmann solution, the only difference being the possible variations of outer quantities with χ,η,τ.

$$u_\perp^{(0)} = \mathbf{V}(\chi,\eta,\tau)\,[1-e^{-\zeta}],\quad P^{(0)} = P^{(0)}(\chi,\eta,\tau),\quad \phi^{(0)} = \phi^{(0)}(\chi,\eta,\tau),$$

with the constraints $\nabla_\perp . \mathbf{V} = 0$

$$\nabla_\perp \phi^{(0)} = \mathbf{V}\times\mathbf{B}.$$

By introducing these expressions in the small terms of equations (27) we get the second order solution

$$u_\perp^{(1)} = \frac{\zeta}{2}e^{-\zeta}\frac{d\mathbf{V}_\perp}{d\tau}+\left(\frac{e^{-2\zeta}}{3}-\frac{e^{-\zeta}}{3}\right)(\mathbf{V}.\nabla_\perp)\,\mathbf{V}$$

$$u_\perp^{(2)} = \left(\frac{\zeta e^{-\zeta}}{2}+e^{-\zeta}+\zeta-1\right)\nabla_\perp\times(\nabla_\perp\times\mathbf{V}).$$

The velocity z component can then be deduced from the continuity equation in (27)

$$w^{(0)} = [\,-\tfrac{5}{24}+\tfrac{1}{6}e^{-\zeta}+\tfrac{1}{24}e^{-2\zeta}+\tfrac{1}{2}\zeta e^{-\zeta}-\tfrac{1}{4}\zeta e^{-2\zeta}]\,\nabla_\perp.[(\mathbf{V}.\nabla_\perp)\,\mathbf{V}].$$

REFERENCES

ALEMANY, A., MOREAU, R., SULEM, P. L. & FRISCH, U. 1979 *J. de Mécanique*, **18**, 277.

BATCHELOR, G. K. 1969 *Phys. Fluids Suppl.* **12**, II–233.

BRANOVER, H. 1978 *Magnetohydrodynamic Flows in Ducts*. Halsted.

COLIN DE VERDIERE, A. 1980 *Geophys. Astroph. Fluid Dyn.* **15**, 213.

GEL'FGAT, YU. M., KIT, L. G., PETERSON, D. A. & TSINOBER, A. B. 1971 *Magn. Gidrodin.* **3**, 35.

HEISER, W. H. & SHERCLIFF, J. A. 1965 *J. Fluid Mech.* **22**, 701.

HOPFINGER, E. & BROWAND, F. 1982 *Nature* (To appear).

KOLESNIKOV, YU. B. & TSINOBER, A. B. 1974 *Izv. A.N.S.S.S.R. Mekh. Zh. i Gaza* **4**, 176.

LEHNERT, B. 1955 *Quart. Appl. Math.* **12**, 321.

LIELAUSIS, O. 1975 *Atomic Energy Review* **13**, 527.

MOREAU, R. 1978 In *Dynamic Measurements in Unsteady Flows*. Proceedings of Dynamic Flow Conference, pp. 65–79.

PLATNIEKS, I. A. & FREIBERGS, YU. ZH. 1972 *Magn. Gidrodin.* **2**, 29.

ROBERTS, P. H. 1967 *An Introduction to Magnetohydrodynamics*. Longmans.

ROSANT, J. M. 1976 Ecoulements hydromagnetiques turbulents en conduites rectangulaires. Thèse 3e cycle, Grenoble.

SHERCLIFF, J. A. 1965 *A Textbook of Magnetohydrodynamics*. Pergamon.

SOMMERIA, J. 1980 Tendance à la bidimensionalité de la turbulence M.H.D. Thèse 3e cycle, Grenoble.

TATSUMI, T. & YANASE, S. 1981 *J. Fluid Mech.* **110**, 475.

VOTSISH, A. D. & KOLESNIKOV, YU. B. 1976 *Magn. Gidrodin.* **3**, 25 and **3**, 141.